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# On the Efficiency and Stability of Networks

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## Abstract

The paper studies the efficiency of economic networks and the stability of links when players freely choose to form or cut them for their self-interests. Although it is desirable that a network is both efficient and stable, these two objectives are often incompatible. Imposing reasonable requirements on the distributive rule, we will give conditions under which this conflict is avoided.

# 1. Introduction

The paper studies the efficiency of economic networks and the stability of links when players freely choose to form or cut them for their self-interests. We will show that some of the efficient networks are unstable and some of the stable networks are inefficient. However we can give reasonable sufficient conditions under which these two requirements are compatible in fairly general settings. We will also show that in some typical networks the conflict can be resolved by imposing reasonable conditions on parameters characterizing the networks.

Our framework is very similar to that in Jackson and Wolinsky (1996) Emphasis is laid on the compatibility of efficiency and stability of networks. We argue that inefficiency results because players' net benefit from the formation of links conflicts with the social benefit. Other related studies include Aumann and Myerson (1988), Myerson (1977) and Qin (1996).

Some recent studies, including Bolton and Dewartripont (1994), Radner (1992) and Hendrics , Piccione and Tan (1995), aim at explaining structural features of networks. Other studies, including Radner (1993), Sobel (1992), Zamir, Kamien and Tauman(1990), highlight aspects of information processing. Economides(1996a,b) and Economides and White (1994) relate the compatibility and the networks that rely on it with the vertically related industries and discuss policy issues arising in modern economies.

## 2. Key Concepts and Notations

Let $N=\{1,2,\dots,n\}$ be the set of players. The network relations among them are represented by graphs whose nodes are identified with the players and whose arcs indicate the pairwise relations.

The graph representing the set of all subsets of $N$ of size 2 will be called the complete graph and denoted $g^N$. It will also be referred to as a *point-to-point network*. The subset of $N$ containing $i$ and $j$ will be denoted as $ij$. $ij \in g$ means that $i$ and $j$ are directly connected. We let $g+ij$ denote the graph obtained by adding link $ij$ to $g$ and $g-ij$ denote the graph obtained by deleting link $ij$ from $g$.

A sequence of direct connections will be called a *path*. A pair of nodes $i$ and $j$ are *connected* if there is a path between $i$ and $j$. The nodes in $g$ which are connected by a path, together with the corresponding arcs, form a *component* of $g$. The set of components of $g$ will be denoted $C(g)$. A component of a graph will called a *star* or a *hub-spoke network* if all edges are linked to one central edge.

Let $G$ be the set of all subsets (subgraphs) of $g^N$. The performance of the graph $g$ will be captured by a real valued function $v : G \rightarrow R$, often referred to as the social utility function or the valuation function. In some applications the value will be an aggregate of individual utilities or productions and may be expressed as $v(g) = \Sigma_i u_i(g)$ for $g \in G$. The set of all social utility functions will be denoted $V$. *Allocation rule*

$$Y : \mathrm{G} \times \mathrm{V} \rightarrow R^n$$

describes how the value associated with each network are distributed to the individual players. $Y_i(g,v)$ is the payoff to player $i$ from graph $g$ under the value function $v$. A simple example of the allocation rule is the *equal distribution rule* ($\overline{Y}_i = v(g)/n$ for all $i$) which splits the value of the game $v(g)$ equally among the players. A graph $g \in G$ is *strongly efficient* if $v(g) \geq v(g')$ for all $g' \in G$. The term strong efficiency indicates that it refers to the total value rather than Paretian notion.

## 3. Some Examples and Preliminary Remarks

We will discuss whether a network is efficient from a social viewpoint and which links are likely to be formed. It is easy to find examples in which these two objectives are compatible.

Typical examples include the case of a good young couple in love and links among several friendly countries. In these cases the benefit from the connections well compensates for the cost of the formation of links. In the opposite case where the cost is high compared to the benefit, no link will be formed and this will be socially desirable also.

But there are many cases in which the efficiency and stability of networks are incompatible. The basic reason for this is that the individual motive to form or remove links with others may deviate from the social objective. We will discuss that such cases are very typical.

Next two example shows that adding a link may decrease the total utility of the society (even in Paretian sense), even if the cost of formation of network is zero and individuals act so as to maximize their utilities (cf. Garcia, C.B. and W. I. Zangwill (1981)).

<u>Example 1</u>

Suppose there are 3 (thousands) commuters from suburb A to city C and 2 (thousands) commuters from suburb B to city C. It takes 30 minutes to go from A to C, but because of bad road condition, 1 hour is required to go from B to C.

(Fig.1a about here)
(Fig .1b about here)

Because the travel from B to C is slow, it may appear to be desirable to build a new road from B to A. Let $n_1$ (resp.$n_2$) denote the number of the residents of suburb B following the route AC (resp. BC). The cost functions (expressing the loss of hours by commuting) for the suburb A and B are expressed as

$$C_A(n_1) = 30 + n_1$$
$$C_B(n_2) = \min(60, 50 + 2n_2).$$

The case $n_1 = 0$ and $n_2 = 0$ $(C_A = 30, C_B = 60)$ describes the cost functions before the construction of the route.

The equilibrium condition

$$60 = 50 + 2n_2$$

gives $n_2 = 5$ and $C_A = 35$, $C_B = 60$. Hence no one is better off and some one is worse off by the construction of the route.

The famous Braess Paradox shows that by adding a link everyone is made worse off.

Example 2. (Braess Paradox)

(Fig.2a about here)

(Fig.2b about here)

Fig.2a shows the relevant network and the associate cost functions which depend on the numbers of travelers. Assume that 6 people must travel A to D. In equilibrium, hours required for following the route A BD and route A C D must be the same. Hence, letting $n$ denote the number of the people following the first route we have

$$11n + 50 = 11(6 - n) + 50,$$

which yields $n = 3$. It takes 30 minutes from A to B and 53 minutes from A to C. The total required hours are 83 minutes in both routes.

Let us now add a link from B to C which yields the equilibrium conditions.

$10n_1 + n_5 + 10 = n_3 + 50$ (hours from A to C)

$n_2 + 50 = 10n_4 + n_5 + 10$ (hours from B to D)

$n_1 = n_2 + n_5$ (population at B)

$n_2 + n_4 = 6$ (population at D)

$n_1 + n_3 = 6$ (population at A)

These equations yield $n_1 = 4$, $n_2 = 2$, $n_3 = 2$, $n_4 = 4$, $n_5 = 2$, and the travel from A to D requires 92 minutes for any route.

Example 3 (Two Sided Matching)

There are finite disjoint sets of agents, $M=(m_1,\dots,m_n)$ and $W=(w_1,\dots,w_n)$. Each agent $m_i \in M$ owns $a_i$ units of the first resource and each agent $w_i \in M$ owns $b_i$ units of the second resource. We assume that $a_1 < a_2 <\dots< a_n$ and the $b_1 < b_2 <\dots< b_n$.

In the marriage model, $M$ is the set of men and $W$ is the set of women and $a_i$ and $b_j$ express the amounts of some specific talents they have. Alternatively, we may assume that $M$ and $W$ are the sets of agents with two kinds of resources which could be used to produce a homogeneous product.

Each member of $M$ is matched only to one member of $W$ and the total product of the whole matching is given by

$$u(g)=\Sigma_i f(a_i,b_{j(i)})$$

where $j(i)$ is the agent in $W$ who is matched to agent $i$ in $M$.

In general the maximum of the total product may be attained when a man with high talent is matched with a woman of low talent. For example, let $n=2$ and $a_1=1$, $a_2=2$, $b_1=1$, and $b_2=2$, and production function be defined by $f(1,1)=1$, $f(2,2)=4$, $f(1,2)=f(2,1)=3$. Then we have $f(1,2)+f(2,1)>f(1,1)+f(2,2)$, as claimed.

This matching is considered to be very unstable because the man and the woman with high talents can improve upon the original position, if we assume e.g. the equal distribution rule. This conflict can be resolved in the case where the production function satisfies supermodularity as defined below.

We say that a real valued function $u(a,b)$ is *supermodular* if $a'<a$, $b'<b$ implies that

$$u(a',b')+u(a,b)>u(a',b)+u(a,b').$$

Hence the supermodularity $f$ of implies that if $a_1<a_2$ and $b_1<b_2$ then $f(a_1,b_1)+f(a_2,b_2)>f(a_1,b_2)+f(a_2,b_1)$. Using this relation repeatedly, we see that the maximum of the total product is attained when $j(i)=i$, and the maximal value is given by $u^*(g)=\Sigma_i f(a_i,b_i)$. This matching is considered to be stable under the equal distribution rule in the sense that no one is willing to form a new link. This conclusion will be examined in more detail in the following sections.

## 4. Conditions on Stability and Allocation Rules

To describe which networks are likely to arise, we need to introduce a notion of stability and some additional concepts. We say that graph $g$ is *pairwise stable* with respect to the valuation function $v$ and the distribution rule $Y$ if

(i) for all $ij \in g$ $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$

and

(ii) for all $ij \notin g$ if $Y_i(g,v) < Y_i(g+ij,v)$ then $Y_j(g,v) > Y_j(g+ij,v)$.

We say that $g$ improves upon $g'$ with respect to $v$ and $Y$ if

$$Y_i(g,v) \geq Y_i(g',v), \quad \text{for all } i \in S$$

and $$Y_i(g,v) > Y_i(g',v), \quad \text{for some } i \in S.$$

$g$ is *coalitionally stabel (CS)* if there is no $g' \in g^N$ and the set $S$ of players joined by the graph $g'$ which improve upon $g$.

Let $\pi$ be a permutation on the set of players $N$. We define $g^\pi$ as $g^\pi = \{ij | i = \pi(k), j = \pi(l), kl \in g\}$ and let $v^\pi(g^\pi) = v(g)$.

DEFINITION The allocation rule $Y$ is *anonymous* if, for any permutation of $\pi$, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g,v)$.

This means that the distribution rule depends only on the architectural form of the graph and not on the naming of players.

DEFINITION An allocation rule $Y$ is *balanced* (or *feasible*) if $\Sigma_i Y_i(g,v) = v(g)$ for all $v$ and $g$.

DEFINITION A link $ij$ is *critical* to the graph $g$ if $g - ij$ has more components than $g$ or if it is linked only to $j$ under $g$.

Recall that we defined $C(g)$ to be the set of components of $g$. We now introduce a stronger notion of balancedness.

DEFINITION A value function $v$ is component additive if $v(g) = \Sigma_{h \in C(g)} v(h)$

We also give:

DEFINITION An allocation rule $Y$ satisfies *equal bargaining power* (EBP) if for all $v, g$ and $ij \in g$

$$Y_i(g,v) - Y_i(g - ij, v) = Y_j(g, v) - Y_j(g - ij, v)$$

The basic reason for the incompatibility of efficiency and pairwise stability is that, whereas pairwise stability is attained only through the adjustments of the private benefits of the directly connected of players, the social objective concerns with the benefit of the whole network. In particular, when a critical link $ij$ (such a $A_0B_0$ in Fig.4b) is severed for private benefits, other players will be separated and may incur a huge loss, resulting in the decrease in the social utility .

The following example, which is due to Jackson and Wolinsky (1996) shows explicitly the ranges of parameters in which efficiency and pairwise stability are compatible.

<u>Example 4</u> Connections Model (3 players symmetric case)

We assume that players directly communicate with those whom they are liked . In the symmetric case we consider, the intrinsic value of the direct communication is $w$ and that of the indirect communication is (for simplicity) $w^2$. We also assume that the cost of maintaining the direct link is $c$ (with $w > c$) for each player.

Except for the trivial case where no link is formed, there are three representative cases.

(Fig.3 about here)

We will assume that the allocation rule satisfies equal bargaining power.

case (i) When only one link is formed as in Figure 1, each of the 2 players who are directly linked enjoys the utility of $Y_1(g_1) = Y_2(g_1) = w - c$, and $Y_3(g_1) = 0$. Hence the social utility of the graph is given by $v(g_1) = 2(w - c)$,

case (ii) Figure 2 represents the case of a star. The center is directly linked to the other players and the other players are directly linked to the center and indirectly to the third player. Hence the social utility is given by $v(g_2) = 4w + 2w^2 - 4c$.

We have $Y_1(g_2 - 23) = Y_2(g_2 - 23) = w - c$ and $Y_3(g_2 - 23) = 0$. On the other hand, $Y_1(g_2 - 12) = 0$ and $Y_2(g_2 - 12) = Y_3(g_2 - 12) = w - c$. Also from the equal bargaining power, we have $Y_2(g_2) - (w - c) = Y_1(g_2) - 0 = Y_3(g) - 0$. Since $v(g_2) = Y_1(g_2) + Y_2(g_2) + Y_3(g_2) = 4w + 2w^2 - 4c$, we have $Y_1(g_2) = Y_3(g_2) = w + (2/3)w^2 - c$ and $Y_2(g_2) = 2w + (2/3)w^2$.

case (iii) Figure 3 represents the case of the complete graph (or a circle in the present case). Each player is directly linked to others, hence we have $Y_1(g_3) = Y_2(g_3) = Y_3(g_3) = 2(w - c)$ and $v(g_3) = 6(w - c)$.

Graph $g_2$ is stable if no benefit is obtained from removing any of the links. It is shown that in the range $w - 2w^2/3 < c < w$ the strongly efficient network $g_2$ is uniquely stable, but in the range $w - 2w^2 < c < w - 2w^2/3$ the inefficient network $g_3$ is the only stable one.

## 5 Some General Results

In this section we will bring together some of the salient results on the stability of networks and interpret and discuss the implications of related research work stated in somewhat different settings.

Theorem 1 of Jackson and Wolinsky (1996) says that

Proposition 1

If $n \geq 3$, then there is no allocation rule $Y$ which is anonymous and component balanced and such that for each $v$ at least one strongly efficient graph is pairwise stable.

However, the conflict between efficiency and stability can be avoided if we assume either a special nature of distribution rules or restrict the allowable class of graphs. For example, we know the following (see, Theorem 3 of Jackson and Wolinsky (1996)) :

Proposition 2

*Equal distribution rule* ($\overline{Y}_i(g,v) = v(g)/n$ for all $i$ ) is an anonymous and pairwise stable network.

This allocation rule is not very attractive because it is not sensitive to the changes in the structure of network and is not even individual rational in general. There are distributive rules which satisfy stability and other desirable properties for a broad class of graphs, as we show below.

Let $N=\{1,2,...,n\}$ and $(N,u)$ be a $N$-person game in the coalitional form (characteristic function form). We say that game $(N,u)$ is *superadditive* if

(1) $u(S \cup T) \geq u(S) + u(T)$ for all $S$, $T \subset N$ such that $S \cap T = \phi$.

We also say that game $(N,u)$ is *convex*, if for each $i \in N$ and $S \subset T \subset N - \{i\}$,

(2) $$u(S \cup \{i\}) - u(S) \leq u(T \cup \{i\}) - u(T).$$

This means that the contribution of each player $i$ in a set is not smaller than that in its subset. This condition is known to be equivalent to

(3) $$u(S \cup T) + u(S \cap T) \geq u(S) + u(T) \quad \text{for all } S,\ T \subset N$$

(see, Ichiishi (1983) or Driessen (1988)). This set function is also referred to as *supermodular.* Hence so far as $u(\phi)=0$, any convex game is superadditive.

When a graph $g$ and value function $v(g)$ are given, a cooperative game in the coalitional form can be defined in a natural way. For each subset $S$ of $N$, the coalitional game is defined as

(4) $$U_{v,g}(S)=v(g|S)$$

where $g|S=\{ij\in g:\ i\in S \text{ and } j\in S\}$.

This expresses the characteristic function of the original game which would result if we require that players can only cooperate along links in $g$.

For a graph which is not fully connected, let $C(g)$ be the set of components of $g$. Then under the assumption of component additivity, the above characteristic function may be expressed as

(5) $$U_{v,g}(S)=\Sigma_{h\in C(g|S)}v(h).$$

It easily follows that if the characteristic function satisfies $v(\phi)=0$ and superadditity then $v(\mathrm{T})\geq v(S)$ for all $S\subset \mathrm{T}$. This implies the following:

<u>Proposition 3</u>

If the characteristic function (4) is superadditive then the complete graph $g^N$ is strongly efficient.

It is established (Theorem 4 of Jackson and Wolinsky (1996)) that

<u>Proposition 4</u>

If value function $v$ is component additive, then the unique allocation rule $Y$ which satisfies component balance and EBP is the Shapley value (*1) of the game defined by the characteristic function (4) .

The proof relies on the results in cooperative game theory (see, Myerson(1977)) and uses the above definition of the characteristic function. The solution described in Proposition 4 is also referred to as the *Myerson value.*

No explicit statement is made on the stability of networks in Proposition(3). But based on Propositions 3 and 4, we can establish a stability result:

<u>Proposition 5</u>

If the value function $v$ is component additive and allocation rule $Y$ (which reduces to the Shapley value) satisfies component balance and EBP and if, for each fixed $g$, the associated characteristic function (4) is superadditive then the strongly efficient graph is pairwise stable.

For a proof we note, as Myerson(1977) showed explicitly in the proof of his Theorem, that if the characteristic function (4) is superadditive then EBP rule is pairwise stable. Proposition 3 then completes the proof.

In a different context, Qin (1994), utilizing a result of a potential (*2) game (see, Monderer and Shapley (1996)), established an important stability result. He considered a game of coalition formation in which a strategy for a player $i$ is a set of players whom $i$ wishes to form links and a link between a pair of players is actually formed if both players wish to form it. A payoff function is defined as the allocation rule in our model.

A strategic form game with a potential has a learning property. This means that network forming processes converges to equilibria of the game and the stable equilibria maximizes the potential. The main stability results of Qin (1994) may be restated as in the following propositions.

<u>Proposition 6</u>

The Myerson value is feasible and the cooperation formation game has a potential.

<u>Proposition 7</u>

If the payoff of the coalitional game is defined by the characteristic function (4) then the process of forming links with others result in the attainment of full cooperation (complete graph $g^N$ in our context).

It is known that if the game is convex then the core is not empty and the Shapley value is in the core (See, Ichiishi (1983) or Driessen (1988)). This implies that coalitional stability holds in this case.

The attainment of the Shapley value relies on the assumption of equal bargaining power. If we drop this assumption, similar stability results can be obtained for many solution concepts of the cooperative game such as the kernel, the nucleolus, the bargaining set and $\tau$-value (see, Tijs(1987) or Driessen(1988)), once we know the axioms which characterize the solutions and know that the solution is in the core. Thus coalitional stability can be attained for many distribution rules. Superadditivity of a game is a strong assumption in analyzing the entry problem of modern industries, where firms have large fixed costs and social optimality requires to restrict the entry of firms. Convexity of the game (the super modularity of the value function) is also a strong assumption because it implies the superadditivity.

## 6. Conclusion

In many situations efficiency and stability of networks are incompatible. We argued that this is because new links bring about externality which is not taken into account as benefits of the individual players.

These two objectives are compatible if the distribution rule satisfies equal bargaining power and the game is superadditive. A stronger requirement than the latter is the supermodularity or convexity of game, which gurantees stronger coalitional stability under alternative distribution rules. Supermodualtity of the numerical function as defined in example 3 serves for a similar purpose.

Our analysis in this paper has been conducted in a very abstract setting, with no assymmetries among players. In fact to explain the emergence of a star or complete network was one of our chief objectives. In industries such as telephone and railroad, geographical conditions require some specific network structures such as ones with stars. We hope to discuss these problems and policy issues in a later research.

# Footnote

(1) The Shapley value of the characteristic function game $(N,v)$, denoted $\phi(v)=(\phi_1(v),\dots,\phi_n(v))$, is given by

$$\phi_i(v)=\Sigma_S A(s)(v(S)-v(S-\{i\}) \quad \text{with} \quad A(S)=((s-1)!(n-s)!)/n!\,,$$

where $s$ is the number of players in $S$ and the above summation is over all $S$ which contain $i$.

(2) A *potential* for a strategic form game is a function which maps strategy profiles into real numbers (in the present context the action of forming links) such that, when a player deviates the change in the payoff equals the change in the potential (see Monderer and Shapley (1996)) for a formal definition and details).

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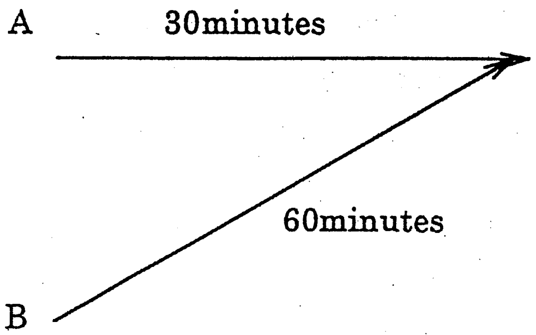
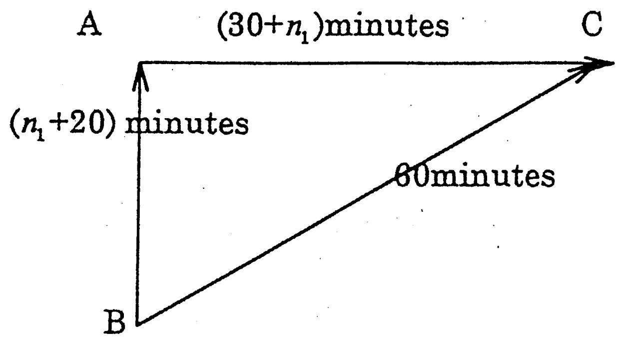



fig. 1a

fig. 1b



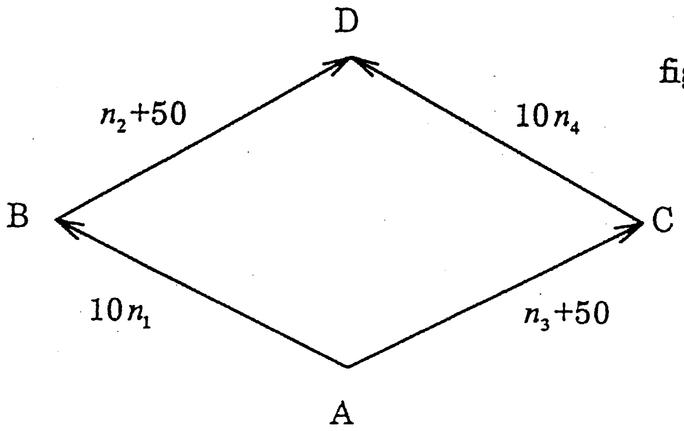
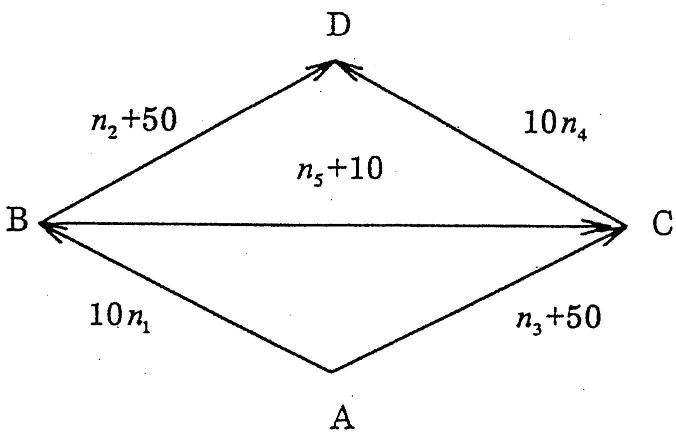




fig. 2a

$n_1=n_2=n_3=n_4=3$

fig. 2b




$n_1=4,\ n_2=2,\ n_3=2,\ n_4=4,\ n_5=2$

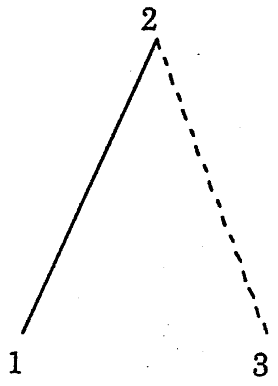



fig. 3a

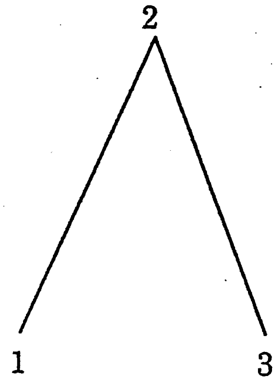



fig. 3b

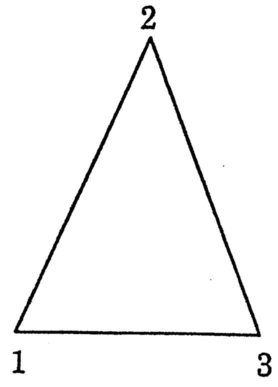



fig. 3c